Finite Orthogonal Geometries with Characteristic $\neq 2$ and PBIB Designs. I

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ABSTRACT

The conjugation relation among the subspaces of a finite orthogonal geometry with characteristic $\neq 2$ and its properties are studied. Then they are used to find relations between some enumeration formulas for the subspaces of the orthogonal geometry, to prove a type of transitivity of the orthogonal groups, to construct PBIB designs and BIB designs, and to establish the isomorphisms between some known PBIB designs.

1. INTRODUCTION

The idea of using vector spaces over finite fields F_q for constructing BIB and PBIB designs originated in the classical papers of R. C. Bose [1, 2] and R. C. Bose and K. R. Nair [3]. Finite geometries were also used by E. J. Primrose [7]. D. K. Ray-Chaudhuri [9], and R. C. Bose and I. M. Chakravarti

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[4] for constructing quadrics and PBIB designs. For references, also see P. Dembowski [6] and D. Raghavarao [8]. In the present paper some properties of finite orthogonal geometries with characteristic $\neq 2$ will be proved and then used for constructing new types of PBIB and BIB designs and for establishing isomorphisms between some known PBIB designs.

Let q be an odd prime power, and F_q the finite field with q elements. It is well known (see, e.g., Chapter 4 of [15]) that the congruent normal forms for the symmetric matrices over F_q are the following:

type $(m, 2s, s)$:

$$\begin{bmatrix} 0 & I^{(s)} & \\ I^{(s)} & 0 & \\ & & 0^{(m-2s)} \end{bmatrix} = M(m, 2s, s),$$

type $(m+1, 2s+1, s, 1)$:

$$\begin{bmatrix} 0 & I^{(s)} & & \\ I^{(s)} & 0 & & \\ & & 1 & \\ & & & 0^{(m-2s)} \end{bmatrix} = M(m+1, 2s+1, s, 1),$$

type $(m+1, 2s+1, s, z)$:

$$\begin{bmatrix} 0 & I^{(s)} & & \\ I^{(s)} & 0 & & \\ & & z & \\ & & & 0^{(m-2s)} \end{bmatrix} = M(m+1, 2s+1, s, z),$$

type $(m+2, 2s+2, s)$:

$$\begin{bmatrix} 0 & I^{(s)} & & & \\ I^{(s)} & 0 & & & \\ & & 1 & & \\ & & & -z & \\ & & & & 0^{(m-2s)} \end{bmatrix} = M(m+2, 2s+2, s),$$

where z is any fixed nonsquare element in F_q. In particular, a nonsingular symmetric $(2\nu+\delta)\times(2\nu+\delta)$ matrix must be congruent to one of the following:

$$S_1=\begin{bmatrix} 0 & I^{(\nu)} \\ I^{(\nu)} & 0 \end{bmatrix} \qquad (\delta=0),$$

$$S_2=\begin{bmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & 1 \end{bmatrix} \qquad (\delta=1),$$

$$S_3=\begin{bmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & z \end{bmatrix} \qquad (\delta=1), \tag{1}$$

$$S_4=\begin{bmatrix} 0 & I^{(\nu)} & & \\ I^{(\nu)} & 0 & & \\ & & 1 & \\ & & & -z \end{bmatrix} \qquad (\delta=2).$$

The orthogonal group of degree $2\nu+\delta$ defined by S_i, where (and throughout this paper) $i=1$; 2 or 3; 4 when $\delta=0$; 1; 2, respectively, is the group whose elements are the matrices T of order $2\nu+\delta$ over F_q satisfying

$$TS_iT'=S_i,$$

and whose operation is the multiplication of matrices, where T' is the transpose of T. This group is usually written as $O_{2\nu+\delta}(F_q,S_i)$.

Let $V_n(F_q)$ be the n-dimensional vector space over F_q. When $n=2\nu+\delta$, the orthogonal group $O_{2\nu+\delta}(F_q,S_i)$ can be viewed as a transformation group of $V_{2\nu+\delta}(F_q)$. The space $V_{2\nu+\delta}(F_q)$ with $O_{2\nu+\delta}(F_q,S_i)$ as its transformation group is called a $(2\nu+\delta)$-dimensional orthogonal space or geometry, and denoted by $V_{2\nu+\delta}(F_q,S_i)$.

Let P be an $m\times(2\nu+\delta)$ matrix with rank m over F_q. We will also use the same symbol P to denote the m-dimensional subspace that the matrix P represents, according to the context. A subspace Q is called a subspace of type $(m+\tau,2s+\tau,s,\Gamma;2\nu+\delta,\Delta)$ (with respect to S_i) if the congruent normal form of QS_iQ' is $M(m+\tau, 2s+\tau, s, \Gamma)$, where

Γ = the definite sign part of the congruent normal form, i.e., Γ is an empty matrix when $\tau = 0$, $\Gamma = 1$ or z when $\tau = 1$, and

$$\Gamma = \begin{bmatrix} 1 & \\ & -z \end{bmatrix} \quad \text{when} \quad \tau = 2;$$

Δ = the definite sign part of S_i, i.e., Δ is an empty matrix when $\delta = 0$, $\Delta = 1$ or z when $\delta = 1$, and

$$\Delta = \begin{bmatrix} 1 & \\ & -z \end{bmatrix} \quad \text{when} \quad \delta = 2$$

$$(1 \leqslant i \leqslant 4).$$

Dai and Feng [5] and Wan, Dai, Feng, and Yang [15] computed the number [written as $N(m+\tau, 2s+\tau, s, \Gamma; 2\nu+\delta, \Delta)$] of subspaces of type $(m+\tau, 2s+\tau, s, \Gamma; 2\nu+\delta, \Delta)$ in the $(2\nu+\delta)$-dimensional orthogonal geometry over F_q, and the number [written as $N(m_1+\tau_1, 2s_1+\tau_1, s_1, \Gamma_1; m+\tau, 2s+\tau, s, \Gamma; 2\nu+\delta, \Delta)$] of subspaces of type $(m_1+\tau_1, 2s_1+\tau_1, s_1, \Gamma_1; 2\nu+\delta, \Delta)$ that are included in a given subspace of type $(m+\tau, 2s+\tau, s, \Gamma; 2\nu+\delta, \Delta)$. For convenience of reference, we restate their results as follows.

THEOREM 1. *When the condition*

$$2s \leqslant m \leqslant \begin{cases} \nu + s - \max(\tau - \delta, 0) & \text{if} \quad \delta \neq \tau \ \text{or} \ \delta = \tau \ \text{and} \ \Gamma = \Delta, \\ \nu + s - 1 & \text{if} \quad \delta = \tau = 1 \ \text{and} \ \Gamma \neq \Delta \end{cases} \tag{2}$$

holds, we have

$$N(m+\tau, 2s+\tau, s, \Gamma; 2\nu+\delta, \Delta)$$
$$= \frac{\prod_{i=\nu+s-m+1}^{\nu} (q^i - 1)(q^{i+\delta-1}+1)}{\prod_{i=1}^{s}(q^i-1) \prod_{i=0}^{s+\tau-1}(q^i+1) \prod_{i=1}^{m-2s}(q^i-1)}$$
$$\cdot q^{2s(\nu+s-m)+s(\delta-\tau)-\tau(m-2s)} n_0(m+\tau, 2s+\tau, s, \Gamma; 2\nu+\delta, \Delta);$$

otherwise, we have

$$N(m+\tau, 2s+\tau, s, \Gamma; 2\nu+\delta, \Delta) = 0,$$

where

$$n_0(m, 2s, s; 2\nu+\delta, \Delta) = 1,$$

$$n_0(m+1, 2s+1, s, \Gamma; 2\nu+\delta, \Delta)$$

$$= \begin{cases} (q^{\nu+s-m}-1)q^{\nu-s-1} & \text{if} \quad \delta = 0, \\ (q^{\nu+s-m}-1)q^{\nu-s} & \text{if} \quad \delta = 1 \text{ and } \Gamma \neq \Delta, \\ (q^{\nu+s-m}+1)q^{\nu-s} & \text{if} \quad \delta = 1 \text{ and } \Gamma = \Delta, \\ (q^{\nu+s-m+1}+1)q^{\nu-s} & \text{if} \quad \delta = 2, \end{cases}$$

$$n_0(m+2, 2s+2, s; 2\nu+\delta, \Delta)$$

$$= \begin{cases} (q^{\nu+s-m-1}-1)(q^{\nu+s-m}-1)q^{2(\nu-s)-2} & \text{if} \quad \delta = 0, \\ (q^{\nu+s-m}-1)(q^{\nu+s-m}+1)q^{2(\nu-s)-1} & \text{if} \quad \delta = 1, \\ (q^{\nu+s-m}+1)(q^{\nu+s-m+1}+1)q^{2(\nu-s)} & \text{if} \quad \delta = 2. \end{cases}$$

THEOREM 2. *If the condition* (2) *holds, then*

$$N(m_1+\tau_1, 2s_1+\tau_1, s_1, \Gamma_1; m+\tau, 2s+\tau, s, \Gamma; 2\nu+\delta, \Delta)$$

$$= \sum_k \frac{\prod_{i=s+s_1-m_1+k+1}^{s} (q^i-1)(q^{i+\tau-1}+1) \prod_{i=m-2s-k+1}^{m-2s} (q^i-1)}{\prod_{i=1}^{s_1}(q^i-1) \prod_{i=0}^{s_1+\tau_1-1} (q^i+1) \prod_{i=1}^{m_1-k-2s_1} (q^i-1) \prod_{i=1}^{k} (q^i-1)}$$

$$\times q^{2s_1(s+s_1-m_1+k)+s_1(\tau-\tau_1)-\tau_1(m_1-k-2s_1)+(m-2s-k)(m_1-k+\tau_1)}$$

$$\times n_0(m_1-k+\tau_i, 2s_1+\tau_1, s_1, \Gamma_1; 2s+\tau, \Gamma),$$

where

$$n_0(m_1-k,2s_1,s_1;2s+\tau,\Gamma)=1,$$

$$n_0(m_1-k+1,2s_1+1,s_1,\Gamma_1;2s+\tau,\Gamma)$$

$$=\begin{cases}(q^{s+s_1-m_1+k}-1)q^{s-s_1-1} & \text{if}\quad \tau=0,\\ (q^{s+s_1-m_1+k}+1)q^{s-s_1} & \text{if}\quad \tau=1\ \text{and}\ \Gamma_1=\Gamma,\\ (q^{s+s_1-m_1+k}-1)q^{s-s_1} & \text{if}\quad \tau=1\ \text{and}\ \Gamma_1\neq\Gamma,\\ (q^{s+s_1-m_1+k+1}+1)q^{s-s_1} & \text{if}\quad \tau=2,\end{cases}$$

$$n_0(m_1-k+2,2s_1+2,s_1,\Gamma_1;2s+\tau,\Gamma)$$

$$=\begin{cases}(q^{s+s_1-m_1+k-1}-1)(q^{s+s_1-m_1+k}-1)q^{2(s-s_1)-2} & \text{if}\quad \tau=0,\\ (q^{s+s_1-m_1+k}-1)(q^{s+s_1-m_1+k}+1)q^{2(s-s_1)-1}, & \text{if}\quad \tau=1,\\ (q^{s+s_1-m_1+k}+1)(q^{s+s_1-m_1+k+1}+1)q^{2(s-s_1)}, & \text{if}\quad \tau=2,\end{cases}$$

and k runs over the range

$$\min(m-2s,m_1-2s_1)\geqslant k$$

$$\geqslant\begin{cases}\max(0,-s-s_1+m_1+\max(\tau_1-\tau,0))\\ \quad \text{if}\quad \tau_1\neq\tau\ \text{or}\ \tau_1=\tau\ \text{and}\ \Gamma_1=\Gamma,\\ \max(0,-s-s_1+m_1+1)\\ \quad \text{if}\quad \tau_1=\tau=1\ \text{and}\ \Gamma_i\neq\Gamma.\end{cases} \tag{3}$$

Note that in these theorems and in the sequel we use the usual convention $\Pi_{i\in\varnothing}N_i=1$ and $\Sigma_{i\in\varnothing}N_i=0$.

Two vectors α and β of the orthogonal geometry $V_{2\nu+\delta}(F_q,S_i)$ are said to be orthogonal (with respect to S_i) if $\alpha S_i\beta'=0$. For an m-dimensional subspace P of $V_{2\nu+\delta}(F_q, S_i)$, we denote by P^* the set of vectors of $V_{2\nu+\delta}(F_q, S_i)$ that are orthogonal (with respect to S_i) to all the vectors of P. Clearly, P^* is a $(2\nu+\delta-m)$-dimensional subspace of $V_{2\nu+\delta}(F_q, s_i)$; it is called the conjugate subspace (with respect to S_i) of P in $V_{2\nu+\delta}(F_q, S_i)$ (also called the conjugation of P).

Dai and Feng [5], Wan, Dai, Feng, and Yang [15], and Shen [11–14] have studied some types of transitivity of the orthogonal groups over F_q, and constructed a number of PBIB designs and BIB designs using the enumeration formulas in Theorems 1 and 2. In the present paper, we will study the properties of the conjugation relation; then use them to find relations between some enumeration formulas, to prove a new type of transitivity of the orthogonal groups, and to construct some new PBIB designs and a BIB design; and finally to point out the isomorphisms between some known PBIB designs.

The concepts and notation used but not defined in this paper are all adopted from [15].

2. THE PROPERTIES OF THE CONJUGATION RELATION

In the following, the conjugation is always with respect to S_i for some i $(1 \leqslant i \leqslant 4)$.

Let P and Q be two subspaces of $V_{2\nu+\delta}(F_q, S_i)$. Then $P \cap Q$ will denote the intersection of P and Q, and $P \cup Q$ the subspace spanned by P and Q.

THEOREM 3. *Let P and Q be two subspaces of $V_{2\nu+\delta}(F_q, S_i)$, and $T \in O_{2\nu+\delta}(F_q, S_i)$. Then*

(a) $(P^*)^* = P$;
(b) *if* $P \subseteq Q$, *then* $Q^* \subseteq P^*$;
(c) $(PT)^* = P^*T$;
(d) $(P \cup Q)^* = P^* \cap Q^*$;
(e) $(P \cap Q)^* = P^* \cup Q^*$.

The proof is similar to that for the unitary geometry over F_q (see [16]), and so is omitted.

Now we examine the conjugation relation among the subspaces of $V_{2\nu+\delta}(F_q, S_i)$. There are four cases to be considered.

Case 1: $n = 2\nu$, $\delta = 0$, and the orthogonal group is $O_{2\nu}(F_q, S_1)$.

Consider the 2ν-dimensional orthogonal geometry $V_{2\nu}(F_q, S_1)$. Let $2s \leqslant m \leqslant \nu + s$, and

$$P = \begin{array}{c} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} m-s \\ s \end{array} \\ \begin{array}{cccc} m-s & \nu-m+s & s & \nu-s \end{array} \end{array}$$

$$P_1 = \begin{array}{c} \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \nu-s \\ \nu-m+s \end{array} \\ \begin{array}{cccc} s & \nu-s & m-s & \nu-m+s \end{array} \end{array} .$$

Clearly, P is a subspace of type $(m, 2s, s; 2\nu)$, P_1 a subspace of type $(2\nu - m, 2(\nu - m + s), \nu - m + s; 2\nu)$, and $PS_1 P_1' = 0$. Then $P_1 \subseteq P^*$. On the other hand, $\dim P_1 = 2\nu - m = \dim P^*$. Hence $P_1 = P^*$.

Let Q be a subspace of type $(m, 2s, s; 2\nu)$. Since the orthogonal group $O_{2\nu}(F_q, S_1)$ acts transitively on the set of subspaces of the same type, there exists $T \in O_{2\nu}(F_q, S_1)$ such that $Q = PT$. By (c) of Theorem 3, $Q^* = P^*T$. Then Q^* is a subspace of type $(2\nu - m, 2(\nu - m + s), \nu - m + s; 2\nu)$. This proves that the conjugation of a subspace of type $(m, 2s, s; 2\nu)$ is a subspace of type $(2\nu - m, 2(\nu - m + s), \nu - m + s; 2\nu)$.

In the following, when we investigate the conjugation of a subspace of a certain type, we will use the above reasoning, i.e., first constructing the conjugation of a special subspace P of that type, then deriving the type of the conjugation of any subspace of the same type by the transitivity of the orthogonal groups and by (c) of Theorem 3. We will no longer explain this.

Let $2s \leqslant m \leqslant \nu + s - 1$, and

$$P = \begin{array}{c} \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2^{-1}\Gamma \end{array}\right] \begin{array}{l} m-s \\ s \\ 1 \end{array} \\ \begin{array}{cccccc} m-s & \nu-m+s-1 & 1 & s & \nu-s-1 & 1 \end{array} \end{array}$$

$$P_1 = \begin{array}{c} \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2^{-1}\Gamma \end{array}\right] \begin{array}{l} \nu-s-1 \\ \nu-m+s-1 \\ 1 \end{array} \\ \begin{array}{cccccc} s & \nu-s-1 & 1 & m-s & \nu-m+s-1 & 1 \end{array} \end{array}$$

$\Gamma = 1$ or z.

Clearly, P is a subspace of type $(m + 1, 2s + 1, s, \Gamma; 2\nu)$, and $P_1 = P^*$. Therefore, when -1 is a square element of F_q, the conjugation of a subspace of type $(m+1, 2s+2, s, \Gamma; 2\nu)$ is a subspace of type $((2\nu-m-2)+1, 2(\nu-m+s-1)+1, \nu-m+s-1, \Gamma; 2\nu)$; and when -1 is a nonsquare element of F_q, the conjugation of a subspace of type $(m+1, 2s+1, s, 1; 2\nu)$ is a subspace of type $((2\nu-m-2)+1, 2(\nu-m+s-1)+1, \nu-m+s-1, z; 2\nu)$, and the conjugation of a subspace of type $(m+1, 2s+1, s, z; 2\nu)$ is a subspace of type $((2\nu-m-2)+1, 2(\nu-m+s-1)+1, \nu-m+s-1, 1; 2\nu)$.

Let $2s \leqslant m \leqslant \nu+s-2$, and

$$P = \begin{array}{cl} \left[\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2^{-1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2^{-1}z \end{matrix}\right] & \begin{matrix} m-s \\ s \\ 1 \\ 1 \end{matrix} \\ \begin{matrix} m-s & \nu-m+s-2 & 1 & 1 & s & \nu-s-2 & 1 & 1 \end{matrix} & \end{array}$$

$$P_1 = \begin{array}{cl} \left[\begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2^{-1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2^{-1}z \end{matrix}\right] & \begin{matrix} \nu-s-2 \\ \nu-m+s-2 \\ 1 \\ 1 \end{matrix} \\ \begin{matrix} s & \nu-s-2 & 1 & 1 & m-s & \nu-m+s-2 & 1 & 1 \end{matrix} & \end{array}$$

Clearly, P is a subspace of type $(m+2, 2s+2, s, 2\nu)$, and $P_1 = P^*$ is a subspace of type $((2\nu-m-4)+2, 2(\nu-m+s-2)+2, \nu-m+s-2; 2\nu)$. Therefore, the conjugation of a subspace of type $(m+2, 2s+2, s; 2\nu)$ is a subspace of type $((2\nu-m-4)+2, 2(\nu-m+s-2)+2, \nu-m+s-2; 2\nu)$.

Case 2: $n = 2\nu+1$, $\delta = 1$, and the orthogonal group is $O_{2\nu+1}(F_q, S_2)$

Consider the orthogonal geometry $V_{2\nu+1}(F_q, S_2)$. Let $2s \leqslant m \leqslant \nu+s$, and

$$P = \begin{array}{cl} \left[\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{matrix}\right] & \begin{matrix} m-s \\ s \end{matrix} \\ \begin{matrix} m-s & \nu-m+s & s & \nu-s & 1 \end{matrix} & \end{array}$$

$$P_1 = \begin{array}{cl} \left[\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}\right] & \begin{matrix} \nu-s \\ \nu-m+s \\ 1 \end{matrix} \\ \begin{matrix} s & \nu-s & m-s & \nu-m+s & 1 \end{matrix} & \end{array}$$

Clearly, P is a subspace of type $(m, 2s, s; 2\nu+1, 1)$, and $P_1 = P^*$ is a subspace of type $((2\nu-m)+1, 2(\nu-m+s)+1, \nu-m+s, 1; 2\nu+1, 1)$. Therefore, the conjugation of a subspace of type $(m, 2s, s; 2\nu+1, 1)$ is a subspace of type $((2\nu-m)+1, 2(\nu-m+s)+1, \nu-m+s, 1; 2\nu+1, 1)$. By

(a) of Theorem 3, the conjugation of a subspace of type $(m+1, 2s+1, s, 1; 2\nu+1, 1)$ is a subspace of type $(2\nu-m, 2(\nu-m+s), \nu-m+s; 2\nu+1, 1)$.

Let $2s \leqslant m \leqslant \nu+s-1$, and

$$P = \begin{array}{c} \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2^{-1}z & 0 \end{array}\right] \begin{array}{l} m-s \\ s \\ 1 \end{array} \\ \begin{array}{ccccccc} m-s & \nu-m+s-1 & 1 & s & \nu-s-1 & 1 & 1 \end{array} \end{array}$$

$$P_1 = \begin{array}{c} \left[\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2^{-1}z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \nu-s-1 \\ \nu-m+s-1 \\ 1 \\ 1 \end{array} \\ \begin{array}{ccccccc} s & \nu-s-1 & 1 & m-s & \nu-m+s-1 & 1 & 1 \end{array} \end{array}$$

Clearly, P is a subspace of type $(m+1, 2s+1, s, z; 2\nu+1, 1)$, and $P_1 = P^*$ is a subspace of type

$$((2\nu-m-2)+2, 2(\nu-m+s-1)+2, \nu-m+s-1; 2\nu+1, 1).$$

Therefore, the conjugation of a subspace of type $(m+1, 2s+1, s, z; 2\nu+1, 1)$ is a subspace of type $((2\nu-m-2)+2, 2(\nu-m+s-1)+2, \nu-m+s-1; 2\nu+1, 1)$. By (a) of Theorem 3, the conjugation of a subspace of type $(m+2, 2s+2, s; 2\nu+1, 1)$ is a subspace of type $((2\nu-m-2)+1, 2(\nu-m+s-1)+1, \nu-m+s-1, z; 2\nu+1, 1)$.

Case 3: $n = 2\nu+1$, $\delta = 1$, and the orthogonal group is $O_{2\nu+1}(F_q, S_3)$

Consider the orthogonal geometry $V_{2\nu+1}(F_q, S_3)$. In a similar way, it follows that when $2s \leqslant m \leqslant \nu+s$, the conjugation of a subspace of type $(m, 2s, s; 2\nu+1, z)$ is a subspace of type $((2\nu-m)+1, 2(\nu-m+s)+1, \nu-m+s, z; 2\nu+1, z)$, and the conjugation of a subspace of type $(m+1, 2s+1, s, z; 2\nu+1, z)$ is a subspace of type $(2\nu-m, 2(\nu-m+s), \nu-m+s; 2\nu+1, z)$; and when $2s \leqslant m \leqslant \nu+s-1$, the conjugation of a subspace of type $(m+1, 2s+1, s, 1; 2\nu+1, z)$ is a subspace of type $((2\nu-m-2)+2, 2(\nu-m+s-1)+2, \nu-m+s-1; 2\nu+1, z)$, and the conjugation of a subspace of type $(m+2, 2s+2, s; 2\nu+1, z)$ is a subspace of type $((2\nu-m-2)+1, 2(\nu-m+s-1)+1, \nu-m+s-1, 1; 2\nu+1, z)$.

Case 4: $n = 2\nu + 2$, $\delta = 2$, and the orthogonal group is $O_{2\nu+2}(F_q, S_4)$

Let $2s \leqslant m \leqslant \nu + s$, and

$$P = \begin{array}{cc} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] & \begin{array}{l} m-s \\ s \end{array} \\ \begin{array}{ccccc} m-s & \nu-m+s & s & \nu-s & 2 \end{array} & \end{array}$$

$$P_1 = \begin{array}{cc} \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] & \begin{array}{l} \nu-s \\ \nu-m+s \\ 2 \end{array} \\ \begin{array}{ccccc} s & \nu-s & m-s & \nu-m+s & 2 \end{array} & \end{array}$$

Clearly, P is a subspace of type $(m, 2s, s; 2\nu + 2)$, and $P_1 = P^*$ is a subspace of type $((2\nu - m) + 2, 2(\nu - m + s) + 2, \nu - m + s; 2\nu + 2)$. Therefore, the conjugation of a subspace of type $(m, 2s, s; 2\nu + 2)$ is a subspace of type $((2\nu - m) + 2, 2(\nu - m + s) + 2, \nu - m + s; 2\nu + 2)$. By (a) of Theorem 3, the conjugation of a subspace of type $(m + 2, 2s + 2, s; 2\nu + 2)$ is a subspace of type $(2\nu - m, 2(\nu - m + s), \nu - m + s; 2\nu + 2)$. Let

$$Q = \begin{array}{cc} \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] & \begin{array}{l} m-s \\ s \\ 1 \end{array} \\ \begin{array}{cccccc} m-s & \nu-m+s & s & \nu-s & 1 & 1 \end{array} & \end{array}$$

$$Q_1 = \begin{array}{cc} \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] & \begin{array}{l} \nu-s \\ \nu-m+s \\ 1 \end{array} \\ \begin{array}{cccccc} s & \nu-s & m-s & \nu-m+s & 1 & 1 \end{array} & \end{array}$$

Clearly, Q is a subspace of type $(m + 1, 2s + 1, s, 1; 2\nu + 2)$, and $Q_1 = Q^*$ is a subspace of type $((2\nu - m) + 1, 2(\nu - m + s) + 1, \nu - m + s, z; 2\nu + 2)$ (when -1 is a square element of F_q) or of type $((2\nu - m) + 1, 2(\nu - m + s) + 1, \nu - m + s, 1; 2\nu + 2)$ (when -1 is a nonsquare element of F_q). Then by (a) of Theorem 3, when -1 is a square element of F_q, the conjugation of a subspace of type $(m + 1, 2s + 1, s, 1; 2\nu + 2)$ is a subspace of type $((2\nu - m) + 1, 2(\nu - m + s) + 1, \nu - m + s, z; 2\nu + 2)$, and the conjugation of a subspace of type $(m + 1, 2s + 1, s, z; 2\nu + 2)$ is a subspace of type $((2\nu - m) + 1, 2(\nu - m + s) + 1, \nu - m + s, 1; 2\nu + 2)$; and when -1 is a nonsquare element of F_q, the conjugation of a subspace of type $(m + 1, 2s + 1, s, 1; 2\nu + 2)$ is a subspace of type $((2\nu - m) + 1, 2(\nu - m + s) + 1, \nu - m + s, 1; 2\nu + 2)$. We are now going to determine the type of the conjugation of a subspace of type $(m + 1, 2s + 1, s, z; 2\nu + 2)$ if -1 is a nonsquare element of F_q.

Suppose -1 is a nonsquare element of F_q. When x runs over all the q elements of F_q, x^2+1 takes $\frac{1}{2}(q+1)$ nonzero elements of F_q as its values. Since there are exactly $\frac{1}{2}(q-1)$ square elements in F_q, there exists $b \in F_q$ such that b^2+1 is a nonsquare element. Then $(b^2+1)z$ is a nonzero square element of F_q: $(b^2+1)z = a^2$, where $a \in F_q \setminus \{0\}$. That is,

$$(a \quad b)\begin{bmatrix} 1 & 0 \\ 0 & -z \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = z.$$

On the other hand,

$$(a \quad b)\begin{bmatrix} 1 & \\ & -z \end{bmatrix}\begin{bmatrix} a^{-1}bz \\ 1 \end{bmatrix} = 0,$$

and

$$\begin{aligned}(a^{-1}bz \quad 1)\begin{bmatrix} 1 & \\ & -z \end{bmatrix}\begin{bmatrix} a^{-1}bz \\ 1 \end{bmatrix} &= (a^{-1}bz)^2 - z \\ &= a^{-2}z(b^2z - a^2) = -(a^{-1}z)^2\end{aligned}$$

is a nonsquare element. Therefore, if

$$P = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b \end{bmatrix} \begin{array}{l} m-s \\ s \\ 1 \end{array} \\ \begin{array}{cccccc} m-s & \nu-m+s & s & \nu-s & 1 & 1 \end{array} \end{array}$$

$$P_1 = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a^{-1}bz & 1 \end{bmatrix} \begin{array}{l} \nu-s \\ \nu-m+s \\ 1 \end{array} \\ \begin{array}{cccccc} s & \nu-s & m-s & \nu-m+s & 1 & 1 \end{array} \end{array}$$

then P is a subspace of type $(m+1, 2s+1, s, z; 2\nu+2)$, and $P_1 = P^*$ is a subspace of type $((2\nu-m)+1, 2(\nu-m+s)+1, \nu-m+s, z; 2\nu+2)$. Hence the conjugation of a subspace of type $(m+1, 2s+1, s, z; 2\nu+2)$ is a subspace of type $((2\nu-m)+1, 2(\nu-m+s)+1, \nu-m+s, z; 2\nu+2)$.

In order to simplify our statement, we will use the symbol $[m+\tau, 2s+\tau, s, \Gamma; 2\nu+\delta, \Delta]$ to express a subspace of type $(m+\tau, 2s+\tau, s, \Gamma; 2\nu$

$+\delta, \Delta)$, and use the expression

$$[m+\tau, 2s+\tau, s, \Gamma; 2\nu+\delta, \Delta]^* = [m_1+\tau_1, 2s_1+\tau_1, s_1, \Gamma_1; 2\nu+\delta, \Delta]$$

to express that the conjugation of a subspace of type $(m+\tau, 2s+\tau, s, \Gamma; 2\nu+\delta, \Delta)$ is a subspace of type $(m_1+\tau_1, 2s_1+\tau_1, s_1, \Gamma_1; 2\nu+\delta, \Delta)$. Furthermore, we define

$$\overline{\Gamma} = \begin{cases} 1 & \text{if} \quad \Gamma = z, \\ z & \text{if} \quad \Gamma = 1, \end{cases}$$

$$\overline{\Delta} = \begin{cases} 1 & \text{if} \quad \Delta = z, \\ z & \text{if} \quad \Delta = 1. \end{cases}$$

Summarizing the above facts, we obtain

THEOREM 4.

(1) *In the orthogonal geometry* $V_{2\nu}(F_q, S_1)$, *we have*

(a) *If* $2s \leqslant m \leqslant \nu + s$, *then*

$$[m, 2s, s; 2\nu]^* = [2\nu - m, 2(\nu - m + s), \nu - m + s; 2\nu].$$

(b) *If* $2s \leqslant m \leqslant \nu + s - 1$, *then*

$$[m+1, 2s+1, s, \Gamma; 2\nu]^*$$

$$= \begin{cases} [(2\nu - m - 2) + 2, 2(\nu - m + s - 1) + 1, \nu - m + s - 1, \Gamma; 2\nu] \\ \quad \text{if } -1 \text{ is a square element of } F_q, \\ [(2\nu - m - 2) + 1, 2(\nu - m + s - 1) + 1, \nu - m + s - 1, \overline{\Gamma}; 2\nu] \\ \quad \text{if } -1 \text{ is a nonsquare element of } F_q. \end{cases}$$

(c) *If* $2s \leqslant m \leqslant \nu + s - 2$, *then*

$$[m+2, 2s+2, s; 2\nu]^*$$

$$= [(2\nu - m - 4) + 2, 2(\nu - m + s - 2) + 2, \nu - m + s - 2; 2\nu].$$

(2) *In both the orthogonal geometry* $V_{2\nu+1}(F_q, S_2)$ *and the orthogonal geometry* $V_{2\nu+1}(F_q, S_3)$, *we have*

(a) *If* $2s \leqslant m \leqslant \nu + s$, *then*

$$[m, 2s, s; 2\nu+1, \Delta]^* = [(2\nu - m)+1, 2(\nu - m + s)+1, \nu - m + s, \Delta; 2\nu+1, \Delta],$$

$$[m+1, 2s+1, s, \Delta; 2\nu+1, \Delta]^* = [2\nu - m, 2(\nu - m + s), \nu - m + s; 2\nu+1, \Delta].$$

(b) *If* $2s \leqslant m \leqslant \nu + s - 1$, *then*

$$[m+1, 2s+1, s, \overline{\Delta}; 2\nu+1, \Delta]^* = [(2\nu - m - 2)+2, 2(\nu - m + s - 1)+2, \nu - m + s - 1; 2\nu+1, \Delta],$$

$$[m+2, 2s+2, s; 2\nu+1, \Delta]^* = [(2\nu - m - 2)+1, 2(\nu - m + s - 1)+1, \nu - m + s - 1, \overline{\Delta}; 2\nu+1, \Delta].$$

(3) *In the orthogonal geometry* $V_{2\nu+2}(F_q, S_4)$, *if* $2s \leqslant m \leqslant \nu + s$, *then*

$$[m, 2s, s; 2\nu+2]^* = [(2\nu - m)+2, 2(\nu - m + s)+2, \nu - m + s; 2\nu+2],$$

$$[m+2, 2s+2, s; 2\nu+2] = [2\nu - m, 2(\nu - m + s), \nu - m + s; 2\nu+2].$$

$$[m+1, 2s+1, s, \Gamma; 2\nu+2]^* = \begin{cases} [(2\nu - m)+1, 2(\nu - m + s)+1, \nu - m + s, \overline{\Gamma}; 2\nu+2] \\ \quad \text{if } -1 \text{ is a square element of } F_q, \\ [(2\nu - m)+1, 2(\nu - m + s)+1, \nu - m + s, \Gamma; 2\nu+2], \\ \quad \text{if } -1 \text{ is a nonsquare element of } F_q. \end{cases}$$

(4) *Clearly, the conjugation mapping between the subspaces of two types in each of the above expressions is a one-one mapping.*

3. SOME APPLICATIONS OF CONJUGATION RELATIONS

Denote by $N^T(m_1 + \tau_1, 2s_1 + \tau_1, s_1, \Gamma_1; m + \tau, 2s + \tau, s, \Gamma; 2\nu + \delta, \Delta)$ the number of subspaces of type $(m + \tau, 2s + \tau, s, \Gamma; 2\nu + \delta, \Delta)$ that include a given subspace of type $(m_1 + \tau_1, 2s_1 + \tau_1, s_1, \Gamma_1; 2\nu + \delta, \Delta)$. One can easily find relations between some enumeration formulas for subspaces from Theorem 4.

THEOREM 5. *If*

$$[m + \tau, 2s + \tau, s, \Gamma; 2\nu + \delta, \Delta]^* = [m_1 + \tau_1, 2s_1 + \tau_1, s_1, \Gamma_1; 2\nu + \delta, \Delta],$$

then

$$N(m + \tau, 2s + \tau, s, \Gamma; 2\nu + \delta, \Delta) = N(m_1 + \tau_1, 2s_1 + \tau_1, s_1, \Gamma_1; 2\nu + \delta, \Delta).$$

If the above condition holds and

$$[m_2 + \tau_2, 2s_2 + \tau_2, s_2, \Gamma_2; 2\nu + \delta, \Delta]^* = [m_3 + \tau_3, 2s_3 + \tau_3, s_3, \Gamma_3; 2\nu + \delta, \Delta],$$

then

$$\begin{aligned} &N^T(m + \tau, 2s + \tau, s, \Gamma; m_2 + \tau_2, 2s_2 + \tau_2, s_2, \Gamma_2; 2\nu + \delta, \Delta) \\ &\quad = N(m_3 + \tau_3, 2s_3 + \tau_3, s_3, \Gamma_3; m_1 + \tau_1, 2s_1 + \tau_1, s_1, \Gamma; 2\nu + \delta, \Gamma). \end{aligned}$$

For example, we have

$$N(m, 2s, s; 2\nu) = N(2\nu - m, 2(\nu - m + s), \nu - m + s; 2\nu),$$

$$\begin{aligned} &N^T(m, 2s, s; m + 1, 2s + 1, s, \Gamma; 2\nu) \\ &\quad = N((2\nu - m - 2) + 1, 2(\nu - m + s - 1) + 1, \nu - m + s - 1, \Gamma; \\ &\qquad\qquad 2\nu - m, 2(\nu - m + s), \nu - m + s; 2\nu), \end{aligned}$$

and so forth.

One also can easily find from Theorem 4 some relations among the subspaces of higher dimensions from the relations among the subspaces of lower dimensions, and find the transitivity of the orthogonal groups on the set of subspace pairs of higher dimensions from the transitivity of the orthogonal groups on the set of subspace pairs of lower dimensions; and vice versa. For example, we have

THEOREM 6. *The intersection of two subspaces of type* $((2\nu-1)+\delta, 2(\nu-1)+\delta, \nu-1, \Delta; 2\nu+\delta, \Delta)$ *is a subspace either of type* $((2\nu-2)+\delta, 2(\nu-1)+\delta, \nu-1, \Delta; 2\nu+\delta, \Delta)$ *or of type* $((2\nu-2)+\delta, 2(\nu-2)+\delta, \nu-2, \Delta; 2\nu+\delta, \Delta)$.

Proof. By Theorem 4,

$$[m, 2s, s; 2\nu+\delta, \Delta]^* = [(2\nu-m)+\delta, 2(\nu-m+s)+\delta, \nu-m+s, \Delta; 2\nu+\delta, \Delta].$$

in particular,

$$[1,0,0; 2\nu+\delta, \Delta]^* = [(2\nu-1)+\delta, 2(\nu-1)+\delta, \nu-1, \Delta; 2\nu+\delta, \Delta].$$

We know that the subspace spanned by two subspaces of type $(1,0,0; 2\nu+\delta, \Delta)$ is a subspace either of type $(2,2,1; 2\nu+\delta, \Delta)$ or of type $(2,0,0; 2\nu+\delta, \Delta)$. Since

$$[2,2,1; 2\nu+\delta, \Delta]^* = [2(\nu-1)+\delta, 2(\nu-1)+\delta, \nu-1, \Delta; 2\nu+\delta, \Delta],$$

$$[2,0,0; 2\nu+\delta, \Delta]^* = [(2\nu-2)+\delta, 2(\nu-2)+\delta, \nu-2, \Delta; 2\nu+\delta, \Delta],$$

$$(P \cup Q)^* = P^* \cap Q^*,$$

the conclusion of the theorem follows. ∎

THEOREM 7. *The orthogonal group* $O_{2\nu+\delta}(F_q, S_i)$ *acts transitively on the set of the subspace pairs the two subspaces of each of which are of type* $((2\nu-1)+\delta, 2(\nu-1)+\delta, \nu-1, \Delta; 2\nu+\delta, \Delta)$ *and intersect in a subspace of type* $((2\nu-2)+\delta, 2(\nu-1)+\delta, \nu-1, \Delta; 2\nu+\delta, \Delta)$, *as well as transitively on the set of subspace pairs the two subspaces of each of which are of type* $((2\nu-1)+\delta, 2(\nu-1)+\delta, \nu-1, \Delta; 2\nu+\delta, \Delta)$ *and intersect in a subspace of type* $((2\nu-2)+\delta, 2(\nu-2)+\delta, \nu-2, \Delta; 2\nu+\delta, \Delta)$.

Proof. It is easily proved that the orthogonal group acts transitively on the set of subspace pairs the two subspaces of each of which are of type

$(1,0,0;2\nu+\delta,\Delta)$ and span a subspace of type $(2,2,1;2\nu+\delta,\Delta)$, as well as transitively on the set of subspace pairs the two subspaces of each of which are of type $(1,0,0;2\nu+\delta,\Delta)$ and span a subspace of type $(2,0,0;2\nu+\delta,\Delta)$. Then by Theorem 4 the conclusion of the theorem follows. ■

Using the transitivity of the orthogonal groups, Wan et al. [15] have constructed an association scheme with two associate classes:

THEOREM 8. *Let* $\nu \geqslant 2$, *and* $\delta = 0$, 1, *or* 2. *Let* $V_{2\nu+\delta}(F_q, S_i)$ *be the* $(2\nu+\delta)$*-dimensional orthogonal geometry, where*

$$i = \begin{cases} 1 & \text{if} \quad \delta = 0, \\ 2 \text{ or } 3 & \text{if} \quad \delta = 1, \\ 4 & \text{if} \quad \delta = 2. \end{cases}$$

The subspaces of type $(1,0,0;2\nu+\delta,\Delta)$ *in* $V_{2\nu+\delta}(F_q, S_i)$ *are taken as treatments, and two treatments are defined to be the first (second) associates of each other if as subspaces they span a subspace of type* $(2,0,0;2\nu+\delta,\Delta)$ *(of type* $(2,2,1;2\nu+\delta,\Delta)$*). Then one obtains an association scheme with two associate classes and with the parameters*

$$v = \frac{(q^{\nu}-1)(q^{\nu+\delta-1}+1)}{q-1}, \qquad n_1 = \frac{(q^{\nu-1}-1)(q^{\nu+\delta-2}+1)}{q-1}q,$$

$$p_{11}^1 = (q-1) + \frac{(q^{\nu-2}-1)(q^{\nu+\delta-3}+1)}{q-1}, \qquad p_{11}^2 = \frac{(q^{\nu-1}-1)(q^{\nu+\delta-2}+1)}{q-1}. \tag{4}$$

From the conjugation mapping, we know that the following association scheme is isomorphic to this association.

THEOREM 9. *Let* ν, δ, *and* $V_{2\nu+\delta}(F_q, S_i)$ *have the same meaning as in Theorem* 8. *The subspaces of type* $((2\nu-1)+\delta, 2(\nu-1)+\delta, \nu-1, \Delta; 2\nu+\delta,\Delta)$ *in* $V_{2\nu+\delta}(F_q, S_i)$ *are taken as treatments, and two treatments are defined to be the first (second) associates of each other if as subspaces they intersect in a subspace of type* $((2\nu-2)+\delta, 2(\nu-2)+\delta, \nu-2, \Delta; 2\nu+\delta,\Delta)$ *(of type* $((2\nu-2)+\delta, 2(\nu-1)+\delta, \nu-1, \Delta; 2\nu+\delta,\Delta)$*). Then we obtain an* *association scheme with two associate classes and with the same parameters as in* (4).

Now we use this association scheme to construct a new PBIB design with two associate classes. Its parameters are hard to compute if one uses the association scheme in Theorem 8.

THEOREM 10. *Suppose the subspaces of type* $(3,2,1;4)$ *in* $V_4(F_q,S_1)$ *are taken as treatments, and the association scheme is the one in Theorem* 9. *The subspaces of type* $(2,1,0,1;4)$ *are taken as blocks, and a treatment is defined to be arranged in a block if the two subspaces which are taken as the treatment and the block, respectively, intersect in a subspace of type* $(1,0,0;4)$. *Then we obtain a PBIB design with two associate classes. The parameters of the PBIB design are those in* (4) *with* $\nu=2$, $\delta=0$ *and in the following*:

$$b=\tfrac{1}{2}(q^2-1)(q+1),$$

$$k=2q, \qquad r=(q-1)q,$$

$$\lambda_1=\tfrac{1}{2}(q-1)^2, \qquad \lambda_2=q-1.$$

Proof. The configuration is a PBIB design because of the transitivity of the orthogonal group. By Theorem 1,

$$b=N(2,1,0,1;4)=\tfrac{1}{2}(q^2-1)(q+1).$$

Since k is the number of subspaces of type $(3,2,1;4)$ which intersect a given subspace of type $(2,1,0,1;4)$ in subspaces of type $(1,0,0;4)$, we have

$$\begin{aligned}k&=N(1,0,0;2,1,0,1;4)\\&\quad\times\left(\frac{N(3,2,1;4)N(1,0,0;3,2,1;4)}{N(1,0,0;4)}-\frac{N(3,2,1;4)N(2,1,0,1;3,2,1;4)}{N(2,1,0,1;4)}\right)\\&=2q,\end{aligned}$$

and then

$$r = \frac{bk}{v} = (q-1)q.$$

We now compute λ_1. Let P and Q be two distinct subspaces of type $(3,2,1;4)$, and $D = P \cap Q$ a subspace of type $(2,0,0;4)$. Then λ_1 is the number of subspaces (written as R) of type $(2,1,0,1;4)$ which intersect P as well as Q in subspaces of type $(1,0,0;4)$. We assert that $R \cap P = R \cap Q$. Otherwise, $R \cap P$ and $R \cap Q$ are two distinct 1-dimensional subspaces, so they span a 2-dimensional subspace which is included in R. Then the subspace $(R \cap P) \cup (R \cap Q)$ must be R. But this contradicts the fact that two distinct subspaces of type $(1,0,0;4)$ only span a subspace of either type $(2,0,0;4)$ or type $(2,2,1;4)$. Hence $R \cap P = R \cap Q$ and $R \cap P = R \cap Q \subseteq D$. On the other hand, the number of subspaces of type $(2,1,0,1;4)$ which include a fixed subspace of type $(1,0,0;4)$ that is included in the subspace D is

$$\frac{N(2,1,0,1;4)N(1,0,0;2,1,0,1;4)}{N(1,0,0;4)} = \tfrac{1}{2}(q-1).$$

Furthermore, two subspaces of type $(2,1,0,1;4)$ which include respective distinct subspaces of type $(1,0,0;4)$ of D are distinct, so the number of subspaces of type $(2,1,0,1;4)$ each intersecting D in a subspace of type $(1,0,0;4)$ is

$$N(1,0,0;2,0,0;4)\cdot\tfrac{1}{2}(q-1) = \tfrac{1}{2}(q^2-1).$$

Note that a subspace of type $(2,1,0,1;4)$ in P or in Q intersects D in a subspace of type $(1,0,0;4)$. So each of these subspaces of type $(2,1,0;4)$ has been enumerated once. Therefore, the number of subspaces of type $(2,1,0;4)$ which intersect P as well as Q in subspaces of type $(1,0,0;4)$, i.e. λ_1, is

$$\lambda_1 = \tfrac{1}{2}(q^2-1) - 2N(2,1,0,1;3,2,1;4) = \tfrac{1}{2}(q-1)^2.$$

The value of parameter λ_2 can be easily obtained from $r(k-1) = n_1\lambda_1 + n_2\lambda_2$:

$$\lambda_2 = q-1.$$

This proves the theorem. ■

Using the conjugation, we can construct some more association schemes and PBIB designs, which will appear in our next paper. In addition, we can construct a new BIB design, which is related to a known BIB design.

Let $\nu = 1$. Since there is no subspace of type $(2,0,0;2+\delta)$ in the $(2+\delta)$-dimensional orthogonal geometry, two subspaces of type $(1,0,0;2+\delta,\Delta)$ in this geometry must span a subspace of type $(2,2,1;2+\delta,\Delta)$. And the orthogonal group $O_{2+\delta}(F_q, S_i)$ acts transitively on the set of subspace [of type $(1,0,0;2+\delta,\Delta)$] pairs. Thus a BIB design was obtained and its parameters were computed in [15], when the subspaces of type $(1,0,0;2+\delta,\Delta)$ were taken as treatments, and the subspaces of type $(2,2,1;2+\delta,\Delta)$ were taken as blocks (see Theorem 3 in Chapter 9 of [15]).

Our new BIB design is stated as follows.

THEOREM 11. *In the 4-dimensional orthogonal geometry $V_4(F_q, S_4)$, take subspaces of type $(1,0,0;4)$ as treatments and the subspaces of type $(3,3,1,1;4)$ as blocks, and define a treatment to be arranged in a block if the latter includes the former, both taken as subspaces. Then we obtain a BIB design with the parameters*

$$v = q^2+1, \qquad b = \tfrac{1}{2}q(q^2+1),$$

$$r = \tfrac{1}{2}q(q+1), \qquad k = q+1,\ \lambda = \tfrac{1}{2}(q+1).$$

Proof. By the transitivity of the orthogonal group $O_4(F_q, S_4)$, this is certainly a BIB design. According to Theorems 1 and 2 its parameters are easily computed as follows:

$$v = N(1,0,0;4) = q^2+1,$$

$$b = N(3,3,1,1;4) = \tfrac{1}{2}q(q^2+1),$$

$$k = N(1,0,0;3,3,1,1;4) = q+1,$$

$$r = \frac{bk}{v} = \tfrac{1}{2}q(q+1),$$

$$\lambda = \frac{r(k-1)}{v-1} = \tfrac{1}{2}(q+1).$$

This completes the proof. ■

4. ISOMORPHISMS BETWEEN SOME KNOWN PBIB DESIGNS

In this final section, we will point out that some known PBIB designs are isomorphic to each other.

Theorems 5, 8, 12, and 13 in Chapter 9 of [15] gave the following four PBIB designs, which will be called Designs A, B, C, and D, respectively.

DESIGN A. In the orthogonal geometry $V_4(F_q, S_1)$, take the subspaces of type $(2,0,0;4)$ as treatments, and define two treatments to be the first (the second) associates of each other if as subspaces they intersect in a 1-dimensional (the 0-dimensional) subspace. Take the subspaces of type $(1,0,0;4)$ as blocks, and define a treatment to be arranged in a block if the former includes the latter, both taken as subspaces. Thus, one obtains a PBIB design with two associate classes and with the parameters

$$v = 2(q+1), \qquad b = (q+1)^2, \qquad r = q+1, \qquad k = 2,$$

$$n_1 = q+1, \qquad p_{11}^1 = 0, \qquad p_{11}^2 = q+1, \qquad \lambda_1 = 1, \qquad \lambda_2 = 0. \qquad (5)$$

DESIGN B. The association scheme is the same as in Design A. Take the subspaces of type $(3,2,1;4)$ as blocks, and define a treatment to be arranged in a block if the latter includes the former, both taken as subspaces. Thus, one obtains a PBIB design with two associate classes and with the same parameters as in (5).

DESIGN C. In the 2ν-dimensional orthogonal geometry $V_{2\nu}(F_q, S_1)$, take the subspaces of type $(\nu,0,0;2\nu)$ as treatments, and define two treatments to be the ith associates of each other if as subspaces they span a $(\nu+i)$-dimensional subspace. Take the subspaces of type $(\nu+s, 2s, s; 2\nu)$ as blocks, where $s \leqslant \nu$, and define a treatment to be arranged in a block if the latter includes the former, both taken as subspaces. Thus, one obtains a PBIB design with ν associate classes and with the parameters listed in Theorem 12 of Chapter 9 of [15].

DESIGN D. The association scheme is the same as in Design C. Take the subspaces of type $(m,0,0;2\nu)$ as blocks, where $m \leqslant \nu$, and define a treatment to be arranged in a block if the former includes the latter, both taken as subspaces. Thus, one obtains a PBIB design with ν associate classes and with the parameters listed in Theorem 13 of the same chapter of [15].

For these designs, we can prove

THEOREM 12. *Design A is isomorphic to Design B; and Design D is isomorphic to Design C with* $s = \nu - m$.

Proof. Since

$$[2,0,0;4]^* = [2,0,0;4],$$

$$[1,0,0;4]^* = [3,2,1;4],$$

the conjugation mapping is an isomorphism mapping between Designs A and B by Theorem 3. This proves the first assertion.

Similarly, since

$$[m,0,0;2\nu]^* = [2\nu - m, 2(\nu - m), \nu - m; 2\nu], \qquad m \leqslant \nu,$$

the second assertion follows. This proves the theorem. ■

For the finite symplectic geometries and the finite unitary geometries, there are similar problems and results, which will appear in two future papers.

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